# Demystified Carathéodory extension theorem

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Carathéodory's extension theorem concerns the striking fact that $\sigma$-additivity on an algebra is already enough to extend a set function to a measure on a $\sigma$-algebra containing that algebra.

The theorem, due to the Greek mathematician Constantin Carathéodory, gives a general construction of such an extension. But even Kallenberg, not known for rhetorical excess, calls the key step an "ingenious technical result." The purpose of this note is to arrive at the key step naturally.

Let $\mu$ be a $\sigma$-additive non-negative set function on an algebra $\mathcal{A}$ of subsets of $S$, with $\mu(\varnothing) = 0$ and values in $[0, \infty]$. That is, $\mu$ is a *premeasure*. Let $\mu^*$ be the induced outer measure,

$$\mu^*(T) = \inf \left\{ \sum_{n=1}^{\infty} \mu(A_n) : T \subseteq \bigcup_{n=1}^{\infty} A_n,\ A_n \in \mathcal{A} \right\}.$$

The induced outer measure agrees with $\mu$ on $\mathcal{A}$:

$$\mu^*(A) = \mu(A), \qquad A \in \mathcal{A}.$$

Suppose that we want to adjoin a set $E \subseteq S$ to the algebra. Write

$$\mathcal{A}[E] = \mathrm{alg}(\mathcal{A} \cup \{E\}).$$

The outer measure $\mu^*$ is already defined on every subset of $S$. It is therefore natural to ask whether, on the enlarged algebra $\mathcal{A}[E]$, it is *still* a premeasure extending $\mu$.

**Definition 1.** A set $E \subseteq S$ is *Carathéodory measurable* if

$$\mu^*|_{\mathcal{A}[E]}$$

is a premeasure.

Since $\mu^* = \mu$ on $\mathcal{A}$, such a premeasure automatically extends the one with which we started. Notice also that every $A \in \mathcal{A}$ is measurable in this sense, since $\mathcal{A}[A] = \mathcal{A}$.

We now see what this definition implies for the outer measure itself.

**Lemma 1.** *If $E$ is Carathéodory measurable, then for every $T \subseteq S$,*

$$\mu^*(T) = \mu^*(T \cap E) + \mu^*(T \cap E^c). \tag{1}$$

*Proof.* Since $\mu^*|_{\mathcal{A}[E]}$ is a premeasure, for every $A \in \mathcal{A}$ the disjoint union

$$A = (A \cap E) \cup (A \cap E^c)$$

gives

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c).$$

Now let $T \subseteq S$, and let

$$T \subseteq \bigcup_n A_n, \qquad A_n \in \mathcal{A}.$$

Then

$$\begin{aligned} \mu^*(T \cap E) + \mu^*(T \cap E^c) &\leq \sum_n \mu^*(A_n \cap E) + \sum_n \mu^*(A_n \cap E^c) \\ &= \sum_n \mu^*(A_n) = \sum_n \mu(A_n). \end{aligned}$$

Taking the infimum over all such covers gives

$$\mu^*(T \cap E) + \mu^*(T \cap E^c) \leq \mu^*(T).$$

The reverse inequality follows from subadditivity. □

The splitting identity (1) is different from the definition we started with: it no longer refers to the algebra $\mathcal{A}$, but only to the fixed outer measure $\mu^*$. This suggests considering all sets for which it holds at once. Let

$$\mathcal{M} = \left\{ E \subseteq S : \mu^*(T) = \mu^*(T \cap E) + \mu^*(T \cap E^c) \text{ for every } T \subseteq S \right\}. \tag{2}$$

By Lemma 1, every Carathéodory measurable set belongs to $\mathcal{M}$. In particular, $\mathcal{A} \subseteq \mathcal{M}$. We now ask what structure this class has. The answer is Carathéodory's theorem.

**Theorem 1** (Carathéodory)**.** *The class $\mathcal{M}$ is a $\sigma$-algebra containing $\mathcal{A}$, and $\mu^*|_{\mathcal{M}}$ is a complete measure extending $\mu$. Moreover, $\mathcal{M}$ is exactly the class of Carathéodory measurable sets in the sense of the definition above.*

*Proof.* The class $\mathcal{M}$ is plainly closed under complements. It is also closed under finite unions. Indeed, if $E, F \in \mathcal{M}$, then for every $T \subseteq S$,

$$\begin{aligned} \mu^*(T) &= \mu^*(T \cap E) + \mu^*(T \cap E^c) \\ &= \mu^*(T \cap E) + \mu^*(T \cap E^c \cap F) + \mu^*(T \cap E^c \cap F^c) \\ &\geq \mu^*(T \cap (E \cup F)) + \mu^*(T \cap (E \cup F)^c). \end{aligned}$$

The reverse inequality follows from subadditivity, so $E \cup F \in \mathcal{M}$. Thus $\mathcal{M}$ is an algebra containing $\mathcal{A}$.

Let now $E_n \in \mathcal{M}$ be pairwise disjoint and put

$$E = \bigcup_{n=1}^{\infty} E_n.$$

For every $T \subseteq S$, repeated use of (2) gives

$$\mu^*(T) = \sum_{n=1}^{N} \mu^*(T \cap E_n) + \mu^* \left( T \cap \left( \bigcup_{n=1}^{N} E_n \right)^c \right).$$

Since

$$T \cap E^c \subseteq T \cap \left( \bigcup_{n=1}^{N} E_n \right)^c,$$

we obtain

$$\mu^*(T) \geq \sum_{n=1}^{N} \mu^*(T \cap E_n) + \mu^*(T \cap E^c).$$

Letting $N \to \infty$ and using countable subadditivity on

$$T \cap E = \bigcup_n (T \cap E_n)$$

gives

$$\mu^*(T) \geq \mu^*(T \cap E) + \mu^*(T \cap E^c).$$

The reverse inequality is subadditivity, so $E \in \mathcal{M}$.

For arbitrary $E_n \in \mathcal{M}$, replace them by the disjoint sequence

$$F_1 = E_1, \qquad F_n = E_n \setminus \bigcup_{k<n} E_k.$$

Since $\mathcal{M}$ is an algebra, $F_n \in \mathcal{M}$, and the union is unchanged. Hence $\mathcal{M}$ is a $\sigma$-algebra.

Now let $E_n \in \mathcal{M}$ be pairwise disjoint and set

$$E = \bigcup_n E_n.$$

Since $E \in \mathcal{M}$, repeated use of (2), with $T = E$, gives

$$\mu^*(E) \geq \sum_{n=1}^{\infty} \mu^*(E_n),$$

while countable subadditivity gives the reverse inequality. Thus $\mu^*|_{\mathcal{M}}$ is a measure. It extends $\mu$ because $\mathcal{A} \subseteq \mathcal{M}$ and $\mu^* = \mu$ on $\mathcal{A}$.

Finally, let $N \in \mathcal{M}$ with $\mu^*(N) = 0$, and let $E \subseteq N$. Then $\mu^*(T \cap E) = 0$ for every $T \subseteq S$. Also $T \cap E^c \subseteq T$, while

$$T = (T \cap E) \cup (T \cap E^c),$$

so

$$\mu^*(T) \leq \mu^*(T \cap E) + \mu^*(T \cap E^c) = \mu^*(T \cap E^c) \leq \mu^*(T).$$

Hence $E \in \mathcal{M}$, and the measure is complete.

It remains to relate $\mathcal{M}$ back to the definition with which we started. Lemma 1 shows that every Carathéodory measurable set belongs to $\mathcal{M}$. Conversely, if $E \in \mathcal{M}$, then, since $\mathcal{M}$ is a $\sigma$-algebra containing both $\mathcal{A}$ and $E$,

$$\mathcal{A}[E] \subseteq \mathcal{M}.$$

The restriction $\mu^*|_{\mathcal{M}}$ is a measure, so its restriction to $\mathcal{A}[E]$ is a premeasure. Thus $E$ is Carathéodory measurable according to the definition above. Hence $\mathcal{M}$ is exactly the class of Carathéodory measurable sets. $\square$